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**STUDIES IN THE METHODOLOGY OF
WEAPONS SYSTEMS EFFECTIVENESS
ANALYSIS—USING THE TECHNIQUES
OF SIMULATION, OPTIMIZATION
AND STATISTICS - PHASE II**

**VOLUME IV. FORMULATION AND OPTIMIZATION
OF WARHEAD KILL PROBABILITIES**

**DEPARTMENT OF CIVIL ENGINEERING
LOUISIANA STATE UNIVERSITY**

TECHNICAL REPORT AFATL-TR-69-156, VOLUME IV

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STUDIES IN THE METHODOLOGY OF WEAPONS SYSTEMS EFFECTIVENESS
ANALYSIS - USING THE TECHNIQUES OF SIMULATION, OPTIMIZATION
AND STATISTICS - PHASE II

Volume IV. Formulation and Optimization of Warhead Kill
Probabilities

Dr. Rodolfo J. Aguilar
Mr. John T. Franques, Jr.

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FOREWORD

This report covers work done during the period September 1968 through August 1969 by Louisiana State University, Baton Rouge, Louisiana, under contract F08635-68-C-0107 with the Air Force Armament Laboratory, Eglin Air Force Base, Florida. Program monitor for the Armament Laboratory was Lt. Jerry L. Edwards (ATAD). Project Director for Louisiana State University was Dr. Adrain E. Johnson, Jr., Department of Chemical Engineering.

The report consists of four volumes as follows: Volume I - MOD6DF Systems Simulation, Volume II - Missile Simulation, Volume III - Effects of Parameter Variations on the Capability of Proportional Navigation Missile Against an Optimally Evading Target in the Horizontal Plane, and Volume IV - Formulation and Optimization of Warhead Kill Probabilities.

Information in this report is embargoed under the Department of State International Traffic In Arms Regulations. This report may be released to foreign governments by departments or agencies of the U. S. Government subject to approval of the Air Force Armament Laboratory (ATAD), Eglin AFB, Florida 32542, or higher authority within the Department of the Air Force. Private individuals or firms require a Department of State export license.

This technical report has been reviewed and is approved.

Thomas P. Christie
THOMAS P. CHRISTIE
Chief, Analysis Division

ABSTRACT

The work conducted at Louisiana State University on a class of problems arising in the evaluation of weapons and in the analysis of the best methods for their use is reported. The destructiveness of a specified weapon against a specified target is measured by the ratio of the number of targets destroyed to the number of weapons used.

This report describes the methods developed for the analytical evaluation of average kill probabilities for single weapons and proposes the optimization of pattern parameters for multi-weapon attacks on discretely defined target complexes for future investigation.

An acceleration algorithm for minimizing a convex objective function subject to linear constraints is also presented in detail.

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SECTION I

INTRODUCTION

This is a report of the work conducted at Louisiana State University on a class of problems arising in the evaluation of weapons and in analyzing the best methods for their use. The effectiveness (destructiveness) of a weapon against a specified target can be measured, at least in part, by the ratio of the number of targets destroyed to the number of weapons used. The number of weapons required to destroy a target depends on two primary factors: the probability that the target is destroyed if the weapon hits it and the probability that the weapon will hit target. If, in addition, weapons are used in groups, the results also depend on the firing pattern employed. The multi-weapon case was not considered by the researchers in the work performed to date. This report concentrates on the methods developed for calculating the probabilities of destroying targets with single weapons.

SECTION II

LETHAL AREA

The simplest case of measuring the destructiveness of a weapon against a target occurs when the weapon must hit the target in order to destroy it, but always destroys the target if it hits it. In this case, the probability that a single weapon will destroy the target is just the probability that it hits a certain area, called the Lethal Area. The size of the lethal area is, consequently, a measure of the destructiveness of the weapon against the target. If the weapon has a proximity fuse, it is no longer necessary to hit the target to destroy it, for the proximity fuse will convert a near miss into a destructive explosion. If the charge is sufficient so that an explosion within the radius of action R of the proximity fuse will destroy the target, then the lethal area is increased to the area included within a curve surrounding the cross section of the target at a distance R from its boundary. In the three-dimensional case, one would not have a lethal area defined in this manner, but a lethal volume instead.

If a single hit is not enough to destroy a target, the probabilities P_i of destroying the target if it is hit $i = 1, 2, \dots, N$ times with probabilities F_i must be determined. In this case, the probability of destroying the target is

$$P_k = \sum_{i=1}^N P_i F_i \quad (\text{II-1})$$

The P_i probabilities are sometimes called damage coefficients. If all the damage coefficients are equal to each other, and the probability of a hit is F for all firings, then, the probability of destroying the target with N weapons is, simply,

$$P_k = 1 - (1 - PF)^N \quad (\text{II-2})$$

In other words, the probability of destroying the target with any one hit is always the same, regardless of how many previous hits have occurred. This may be interpreted as meaning that only when a vital spot is hit will the target be destroyed and that hits elsewhere on the target will only damage, but not destroy, the target. This vital spot hypothesis serves to reduce the number of unknown quantities P_i and F_i to two, P and F , and has been found to give satisfactory results in many cases, such as AA hits on aircraft. When the vital spot theory can be applied, the lethal area of a target is defined as the product of the effective area of the target and the probability P that a hit on this area will destroy the target. For example, the probability of sinking a merchant vessel with a torpedo hit is about $1/3$. Consequently, $P = 1/3$ and the lethal area is $1/3$ of the ship's length.

In real life situations it is necessary to consider the variation of the probability of destroying the target as a function of the coordinates of the point at which the hit is made. In such instances, the damage probability becomes a damage function $p_t(x,y)$ where x and y are coordinates centered on the target in a plane normal to the weapon's trajectory. In this case, the lethal area may be computed as

$$L = \iint_A p_t(x,y) dx dy. \quad (II-3)$$

where the integration is over all the area for which $p_t(x,y) > 0$.

Formulas for one-dimensional targets are obviously derived from (II-3).

Random Bombardment

The lethal area serves to evaluate the destruction level attained by weapons delivered at random over an area A . For any given target within the area A , the probability that a given weapon will hit the element of area $dx dy$ is simply $dx dy/A$. The probability that this weapon will destroy the target is therefore

$$P_k^{(1)} = \iint p_t(x,y) \frac{dx dy}{A} = \frac{L}{A}. \quad (II-4)$$

If N weapons are fired, the probability that a given target is destroyed is

$$P_k^{(N)} = 1 - \left(1 - \frac{L}{A}\right)^N \quad (II-5)$$

However,

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x} \quad (II-6)$$

Therefore, as the number of weapons increases, i.e.,

$$N \rightarrow \infty, \quad (II-7)$$

one can show that,

$$P_k^{(N)} \cong 1 - e^{-NL/A} \quad (II-8)$$

This represents the expected fraction of all targets destroyed in the area A .

Weapons Aimed at Small Targets

If weapons are individually aimed at a target whose dimensions are small when compared to the errors in aiming, the variation over the target in the probability of hitting an element of area $dx dy$ can be neglected and the lethal area is an adequate measure of destructiveness.

The bombing errors x and y along the range and deflection directions are usually assumed to be normally distributed with standard deviations σ_x and σ_y , respectively. Then, the probability of hitting a target element $dx dy$ is

$$p_t(x, y) dx dy = (2\pi \sigma_x \sigma_y)^{-1} \text{EXP} \left\{ -\frac{1}{2} \left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \right] \right\} dx dy. \quad (\text{II-9})$$

Near the target, i.e., at $x = y = 0$,

$$p_t(0, 0) dx dy = (2\pi \sigma_x \sigma_y)^{-1} dx dy. \quad (\text{II-10})$$

Consequently, the probability of destroying the target with a single weapon is

$$\begin{aligned} p_k^{(1)} &= \iint p_t(x, y) f(x, y) dx dy \\ &= \iint \frac{f(x, y) dx dy}{2\pi \sigma_x \sigma_y}, \end{aligned} \quad (\text{II-11})$$

where $f(x, y)$ is the probability density function of hitting the target.

Hence,

$$p_k^{(1)} = \iint \frac{f(x, y) dx dy}{2\pi \sigma_x \sigma_y} \approx \frac{L}{2\pi \sigma_x \sigma_y} \quad (\text{II-12})$$

If N weapons are fired independently at the target, the probability of killing the target is

$$p_k^{(N)} = 1 - \left(1 - \frac{L}{2\pi \sigma_x \sigma_y} \right)^N \quad (\text{II-13})$$

Since by assumption L is small compared to σ_x and σ_y , equation (II-13) becomes

$$p_k^{(N)} \approx 1 - e^{-NL/2\pi \sigma_x \sigma_y}. \quad (\text{II-14})$$

Equation (II-14) is derived following the same reasoning used to obtain equation (II-8).

Weapons Aimed at Large Targets

When the target dimensions are such that the assumptions for small targets are no longer valid, the variation over the target of the probability of hitting an area element $dx dy$ must be considered, and the lethal area concept is no longer useful.

Again, let $f(x, y) dx dy$ be the probability of hitting an area element, then, the probability that the target is destroyed by a single weapon is

$$P_k^{(1)} = \iint p_t(x, y) dx dy \quad (II-15)$$

and further progress hinges on the ability to evaluate this integral. Once elevated, however, one has, once more, the result that the expected number of shots required to destroy the target is $1/P_k^{(1)}$, and the probability of destroying the target with N weapons is

$$P_k^{(N)} = 1 - (1 - P_k^{(1)})^N \quad (II-16)$$

which if $P_k^{(1)}$ is small, can be written, as before,

$$P_k^{(N)} = 1 - e^{-NP_k^{(1)}} \quad (II-17)$$

When $p_t(x, y)$ is constant over the target area, equation (II-15) represents the probability of hitting the target, multiplied by p_t , and, if the probability density function $f(x, y)$ is not too complex, the integration can sometimes be carried out. This, however, occurs seldom in practice, and it is precisely the evaluation, in closed form, of the integral type given by equation (II-15) that has been the primary concern of the researchers involved in Task 301.

Fragment and Blast Sensitive Targets

A fragment and blast sensitive target is one in which the major damage mechanism is due to fragmentation and blast effects rather than to a direct impact by the weapon. In this case the damage function $p_t(x, y)$ is the kill probability for a warhead given that it has detonated at (x, y) ; $f(x, y)$ is the probability density function for the warhead detonating at (x, y) .

In this case the integration is carried over the entire xy plane. It is with integrations of this type that the investigation reported here was primarily concerned.

SECTION III

ANALYTICAL EVALUATION OF AVERAGE KILL PROBABILITY

The evaluation of the average kill probability integral given by equation (II-15) has formerly been accomplished by numerical techniques; programs written by Martin Marietta Company are available for this purpose. Such techniques involve random samplings of the damage function employing Monte Carlo procedures.

In order to investigate the nature of the damage functions, regression analysis was applied to numerically defined damage functions generated by computer programs developed for the Air Force Armament Laboratory by Martin Marietta Company. The regressed functions had to possess the property of having high coefficients of correlation as well as being integrable in closed form when convoluted with the probability density function $f(x, y)$ which was assumed bivariate normal with zero coefficient of correlation, i.e.,

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \text{EXP} \left\{ -\frac{1}{2} \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right] \right\}, \quad (\text{III-1})$$

where μ_x and μ_y are weapon biases due to aiming and ballistic errors in the range and deflection directions, respectively. μ_x and μ_y are the coordinates of the weapon's mean point of impact for a target centered at the origin of the coordinate system.

The damage function $p_t(x, y)$ depends upon the target definition as well as upon the following weapon parameters:

1. Height of Burst
2. Terminal Velocity
3. Elevation Attack Angle
4. Fragment Mass or Masses
5. Impact Pattern Dimensions

The numerically defined damage functions are best described in polar coordinates; for this reason it was necessary to transform all mathematical functions to this system, thereby giving

$$f(r, \theta) = \frac{1}{2\pi\sigma_x\sigma_y} \text{EXP} \left\{ -\frac{1}{2} \left[\left(\frac{r \cos \theta - \mu_x}{\sigma_x} \right)^2 + \left(\frac{r \sin \theta - \mu_y}{\sigma_y} \right)^2 \right] \right\} \quad (\text{III-2})$$

where $x = r \cos \theta$,

and $y = r \sin \theta$.

(III-3)

The damage function which proved best suited for regression and analytical integration is of the form

$$p_t(r, \theta) = \text{EXP} [A(\theta)r^2 + B(\theta)r + C] \quad (\text{III-4})$$

where $A(\theta) = a_1 \theta^2 + a_2 \theta + a_3$,

$$B(\theta) = b_1 \theta^2 + b_2 \theta + b_3, \quad (\text{III-5})$$

and $C = \text{Constant}$

Consequently,

$$\begin{aligned} P_k &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_t(x, y) f(x, y) dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} p_t(r, \theta) f(r, \theta) r dr d\theta \\ &= \eta \int_0^{2\pi} \int_0^{\infty} e^{[\alpha(\theta)r^2 + \beta(\theta)r]} r dr d\theta, \end{aligned} \quad (\text{III-6})$$

where

$$\begin{aligned} \alpha(\theta) &= A(\theta) - \frac{1}{2} \left(\frac{\cos^2 \theta}{\sigma_x^2} + \frac{\sin^2 \theta}{\sigma_y^2} \right) \\ \beta(\theta) &= B(\theta) + \mu_x \cos \theta + \mu_y \sin \theta, \end{aligned} \quad (\text{III-7})$$

$$\text{and } \eta = \frac{1}{2\pi\sigma_x\sigma_y} \text{EXP} \left[C - \frac{1}{2} \mu_x^2 - \frac{1}{2} \mu_y^2 \right].$$

The complexity of the functions required that a Maclaurin series expansion of $p_t(r, \theta) f(r, \theta)$ be generated in order to integrate them in closed form. This was accomplished as described in Appendix I, with the integration along the coordinate axes performed from 0 to r_0 and from 0 to θ_0 ; where r_0 is a limiting value of the radius where the regressed equation

for $p_k(r, \theta)$ applies, and θ_0 is one full cycle (360° angle). Both r_0 and θ_0 must be smaller than unity to speed up convergence. The following result was obtained:

$$P_k = \int_0^{\theta_0} \int_0^{r_0} p_t(r, \theta) f(r, \theta) r dr d\theta =$$

$$= \frac{e^{\{C - \frac{1}{2} \mu_x a^2 - \frac{1}{2} \mu_y a^2\}}}{2\pi \sigma_x \sigma_y} \left\{ \frac{\theta_0 r_0^2}{2} + \sum_{i=1}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^{i-1} \binom{i}{j} \frac{r_0^{i-(j-2)} (\theta_0)^{j+1}}{[i-(j-2)](j+1)} \right. \right.$$

$$\left. \left. g^0(i-j)r, j\theta \right] \right\}, \quad (\text{III-8})$$

where

$$g_{mr, n\theta}^0 = \left[\sum_{k=0}^n \binom{n}{k} \{ g_{(m-1)r, (n-k)\theta}^0 p_{r, k\theta}^0 + \binom{m-1}{1} g_{(m-2)r, (n-k)\theta}^0 \right.$$

$$\left. p_{2r, k\theta}^0 \right] [1 - s_m(0)] + s_m(0) s_n(0), \quad \text{for } m \geq 0, n \geq 0, \quad (\text{III-9})$$

$$p_{mr, n\theta}^0 = [2b_1 s_n(2) + b_2 s_n(1) + b_3 s_m(0)] s_m(1)$$

$$+ [4a_1 s_n(2) + 2a_2 s_n(1) + (2a_3 - \frac{1}{2\sigma_x} - \frac{1}{2\sigma_y}) s_n(0)] s_m(2)$$

$$+ \{ \mu_x \cos(\frac{n\pi}{2}) + \mu_y \cos[(\frac{n-1}{2})\pi] \} s_n(1)$$

$$+ \{ 2^n (\frac{1}{2\sigma_y} - \frac{1}{2\sigma_x}) \cos(\frac{n\pi}{2}) \} s_m(2), \quad (\text{III-10})$$

for $m = 1, 2; n \geq 0$,

and

$$s_k(j) = 1 \text{ for } k = j$$

$$s_k(j) = 0 \text{ for } k \neq j \quad (\text{III-11})$$

Typical output from the regression program is given in Appendix II.

Future Work

The Maclaurin expansion solution is not satisfactory due to its slow convergence to P_k . Nevertheless, the average kill probabilities evaluated following this procedure are more reliable than the numerically computed ones because they are obtained from the consideration of a larger number of sample points and also because of the upgrading of information about the nature of the damage functions achieved by the method developed.

To overcome the disadvantages of slow convergence, the following procedure is proposed for future, continued research in this field:

I. Evaluation of Damage Functions

R. Snow and M. Ryan (Snow, 1968, p.5) has assumed the damage function $p_t(x,y)$ of equation (3) to be of Gaussian form:

$$(\text{Equation A}) \quad p_t(x, y) = D(x, y) = D_0 \exp \left(-D_0 \left[\frac{x^2}{R_{(1)}^2} + \frac{y^2}{R_{(2)}^2} \right] \right)$$

Based upon this assumption they have developed many useful closed form relationships for calculating fractional coverage to areal targets.

Preliminary work on this task indicates that the symmetry suggested by this form may not be supported by the numerical functions, particularly for low elevation angles of attack. Also, for detonations of high height of burst (HOB) which are out of the blast range of the target, the damage function takes on lower values near the origin of coordinates due to the relatively small fragmentation densities in the polar zone nearest the nose of most ballistic weapons.

In order to evaluate the validity of the Gaussian damage function, the following study is proposed.

- 1) Test the hypothesis that the damage function is of the form given in equation A against the alternative that it is not.
- 2) Study modifications of the form given in equation A to explain the unsymmetrical cases described above and test the hypothesis that the damage function is of the form given by equation A against the alternative that it is of the modified form.

II. Optimization

The explicit formulation of damage functions for single weapon single target element cases makes possible the study of more complex cases of multiple weapon attacks on multiple element targets, by integrating the same damage function with different density functions.

One problem that has not formerly been treated is that of optimizing the pattern parameters of a multiweapon attack on a discretely defined target complex. It is proposed that optimization techniques for optimizing these pattern parameters, using damage functions described in Part I above, be studied by this task.

Optimization Algorithm

During the summer of 1969, the principal investigator collaborated with other L.S.U. personnel in teaching a course in Systems Optimization Theory at Eglin Air Force Base. In part as a result of this experience, the principal investigator developed an acceleration algorithm to minimize a convex objective function subject to linear constraints. A paper describing the algorithm has been submitted for publication in the Journal of the Operations Research Society of America and is included in this report as Appendix III. It is anticipated that the algorithm will prove to be relevant to the work for Task 301, because the researchers expect that the average kill probability function will be convex and that the ballistic constraints will be expressible as linear inequalities.

Reference

Snow, R., and Ryan, M., A Simplified Weapons Evaluation Model . Report No. RM-5677-PR, The Rand Corporation, December, 1968.

APPENDIX I

MATHEMATICAL DEVELOPMENT

Consider the average kill probability integral,

$$P_K = \iint_{-\infty}^{\infty} P_t(x, y) f(x, y) dx dy \\ = \iint_0^{2\pi\infty} P_t(r, \theta) f(r, \theta) r dr d\theta ;$$

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \text{EXP} \left\{ -\frac{1}{2} \left[\left(\frac{x-Hx}{\sigma_x} \right)^2 + \left(\frac{y-Hy}{\sigma_y} \right)^2 \right] \right\} ;$$

$$P_t(x, y) \Rightarrow P_t(r, \theta) = \text{EXP} \left\{ [A(\theta)r^2 + B(\theta)r + C] \right\};$$

where

$$A(\theta) = a_1\theta^2 + a_2\theta + a_3 ;$$

$$B(\theta) = b_1\theta^2 + b_2\theta + b_3 ;$$

$$C = \text{Constant};$$

and

$$r^2 = x^2 + y^2 \Rightarrow x = r \cos \theta; y = r \sin \theta .$$

Therefore, $f(r, \theta) = \frac{1}{2\pi\sigma_x\sigma_y} \text{EXP} \left\{ -\frac{1}{2} \left[\left(\frac{r \cos \theta - Hx}{\sigma_x} \right)^2 + \left(\frac{r \sin \theta - Hy}{\sigma_y} \right)^2 \right] \right\};$

but $(r \cos \theta - Hx)^2 = r^2 \cos^2 \theta - 2 Hx r \cos \theta + Hx^2 ,$

and $(r \sin \theta - Hy)^2 = r^2 \sin^2 \theta - 2 Hy r \sin \theta + Hy^2 .$

Hence,
$$-\frac{1}{2} \left[\left(\frac{r \cos \theta - Hx}{\sigma_x} \right)^2 + \left(\frac{r \sin \theta - Hy}{\sigma_y} \right)^2 \right] \\ = -\frac{1}{2} \left\{ \frac{1}{\sigma_x^2} [r^2 \cos^2 \theta - 2 Hx r \cos \theta + Hx^2] \right. \\ \left. + \frac{1}{\sigma_y^2} [r^2 \sin^2 \theta - 2 Hy r \sin \theta + Hy^2] \right\} .$$

$$\begin{aligned}\phi(\theta) &= A(\theta) - \frac{1}{2} \left(\frac{\cos^2 \theta}{\sigma_x^2} + \frac{\sin^2 \theta}{\sigma_y^2} \right), \\ \beta(\theta) &= B(\theta) + H_x \cos \theta + H_y \sin \theta, \\ \gamma &= C - \frac{1}{2} H_x^2 - \frac{1}{2} H_y^2 \Rightarrow \text{CONSTANT.}\end{aligned}$$

Then,

$$\begin{aligned}\int_0^{2\pi} \int_0^\infty \mathcal{P}_t(r, \theta) f(r, \theta) r dr d\theta &= \\ &= (2\pi \sigma_x \sigma_y)^{-1} \int_0^{2\pi} \int_0^\infty e^{[\alpha(\theta)r^2 + \beta(\theta)r + \gamma]} r dr d\theta \\ &= (2\pi \sigma_x \sigma_y)^{-1} e^\gamma \int_0^{2\pi} \int_0^\infty e^{[\alpha(\theta)r^2 + \beta(\theta)r]} r dr d\theta\end{aligned}$$

Let $(2\pi \sigma_x \sigma_y)^{-1} e^\gamma = \gamma$

Then,

$$\begin{aligned}\int_0^{2\pi} \int_0^\infty \mathcal{P}_t(r, \theta) f(r, \theta) r dr d\theta &= \\ &= \gamma \int_0^{2\pi} \int_0^\infty e^{[\alpha(\theta)r^2 + \beta(\theta)r]} r dr d\theta.\end{aligned}$$

Consider a Taylor series expansion of a function of two variables about point (r_0, θ_0) ,

$$g(r, \theta) = \sum_{i=0}^{\infty} \frac{\Delta^i g(r_0, \theta_0)}{i!}$$

$$\Delta = \left[(r - r_0) \frac{\partial}{\partial r} + (\theta - \theta_0) \frac{\partial}{\partial \theta} \right]$$

If $r_0 = \theta_0 = 0$, obtain the Maclaurin expansion

or $g(r, \theta)$ as follows:

$$g(r, \theta) = \sum_{i=0}^{\infty} \frac{\Delta^i g(0,0)}{i!} \quad \text{where the operator}$$

$$\Delta = \left[r \frac{\partial}{\partial r} + \theta \frac{\partial}{\partial \theta} \right]$$

Now consider the function

$$e^{[a(\theta)r^2 + \beta(\theta)r]} = e^{p(r, \theta)}$$

$$\begin{aligned} \text{Where } p(r, \theta) &= (a_1 \theta^2 + a_2 \theta + a_3 - \frac{1}{2\sigma_x^2} \cos^2 \theta - \frac{1}{2\sigma_y^2} \sin^2 \theta) r^2 \\ &\quad + (b_1 \theta^2 + b_2 \theta + b_3 + H_x \cos \theta + H_y \sin \theta) r \\ &= \alpha r^2 + \beta r \end{aligned}$$

$$\text{Let } a_4 = -\frac{1}{2\sigma_x^2}; \quad a_5 = -\frac{1}{2\sigma_y^2};$$

$$b_4 = H_x; \quad b_5 = H_y$$

$$\text{Then, } \frac{\partial p}{\partial r} = p_{(1r)} = 2\alpha r + \beta$$

$$\frac{\partial^2 p}{\partial r^2} = p_{(2r)} = 2\alpha$$

$$\frac{\partial^3 p}{\partial r^3} = p_{(3r)} = 0$$

⋮

$$\frac{\partial^n p}{\partial r^n} = p_{(nr)} = 0 \quad \text{for } n \geq 3$$

$$\frac{\partial \mathcal{P}}{\partial \theta} = \mathcal{P}_{(1\theta)} = (2a_1\theta + a_2 - a_4 2 \cos \theta \sin \theta + a_5 2 \cos \theta \sin \theta)r^2 \\ + (2b_1\theta + b_2 - b_4 \sin \theta + b_5 \cos \theta)r$$

$$= [2a_1\theta + a_2 - (a_4 - a_5) 2 \cos \theta \sin \theta]r^2 \\ + [2b_1\theta + b_2 - b_4 \sin \theta + b_5 \cos \theta]r$$

$$\frac{\partial^2 \mathcal{P}}{\partial \theta^2} = \mathcal{P}_{(2\theta)} = [2a_1 - (a_4 - a_5) 2 (-\sin^2 \theta + \cos^2 \theta)]r^2 \\ + [2b_1 - b_4 \cos \theta - b_5 \sin \theta]r \\ = [2a_1 - 2(a_4 - a_5)(1 - 2 \sin^2 \theta)]r^2 \\ + [2b_1 - b_4 \cos \theta - b_5 \sin \theta]r$$

$$\frac{\partial^3 \mathcal{P}}{\partial \theta^3} = \mathcal{P}_{(3\theta)} = [-2(a_4 - a_5)(-4 \sin \theta \cos \theta)]r^2 \\ + [b_4 \sin \theta - b_5 \cos \theta]r \\ = [8(a_4 - a_5) \sin \theta \cos \theta]r^2 \\ + [b_4 \sin \theta - b_5 \cos \theta]r$$

$$\frac{\partial^4 \mathcal{P}}{\partial \theta^4} = \mathcal{P}_{(4\theta)} = [8(a_4 - a_5)(1 - 2 \sin^2 \theta)]r^2 \\ + [b_4 \cos \theta + b_5 \sin \theta]r$$

$$\frac{\partial^5 \mathcal{P}}{\partial \theta^5} = \mathcal{P}_{(5\theta)} = [-32(a_4 - a_5) \sin \theta \cos \theta]r^2 \\ + [-b_4 \sin \theta + b_5 \cos \theta]r$$

$$\frac{\partial^6 \mathcal{P}}{\partial \theta^6} = \mathcal{P}_{(6\theta)} = [-32(a_4 - a_5)(1 - 2 \sin^2 \theta)]r^2 \\ + [-b_4 \cos \theta - b_5 \sin \theta]r$$

$$\frac{\partial^7 \rho}{\partial \theta^7} = \rho_{(7\theta)} = [128(a_4 - a_5)(\sin \theta \cos \theta)]r^2 \\ + [b_4 \sin \theta - b_5 \cos \theta]r$$

$$\frac{\partial^8 \rho}{\partial \theta^8} = \rho_{(8\theta)} = [128(a_4 - a_5)(1 - 2 \sin^2 \theta)]r^2 \\ + [b_4 \cos \theta + b_5 \sin \theta]r$$

⋮

$$\frac{\partial^n \rho}{\partial \theta^n} = \rho_{(n\theta)} = [2^{(n-1)}(a_4 - a_5)(1 - 2 \sin^2 \theta)]r^2 \\ + [b_4 \cos \theta + b_5 \sin \theta]r$$

$$\text{for } n = 4, 8, 12, 16, 20, \dots$$

$$\frac{\partial^n \rho}{\partial \theta^n} = \rho_{(n\theta)} = [-2^{(n-1)}(a_4 - a_5)(1 - 2 \sin^2 \theta)]r^2 \\ + [-b_4 \cos \theta - b_5 \sin \theta]r$$

$$\text{for } n = 6, 10, 14, 18, 22, \dots$$

$$\frac{\partial^n \rho}{\partial \theta^n} = \rho_{(n\theta)} = [-2^n(a_4 - a_5) \sin \theta \cos \theta]r^2 \\ + [-b_4 \sin \theta + b_5 \cos \theta]r$$

$$n = 5, 9, 13, 17, 21, \dots$$

$$\frac{\partial^n \rho}{\partial \theta^n} = \rho_{(n\theta)} = [2^n(a_4 - a_5) \sin \theta \cos \theta]r^2 \\ + [b_4 \sin \theta - b_5 \cos \theta]r$$

$$\text{for } n = 3, 7, 11, 15, 19, 23, \dots$$

Further,

$$\frac{\partial^2 \mathcal{P}}{\partial \theta^n} = \mathcal{P}_{(n\theta)} = (-1)^{\frac{n-4}{2}} \left\{ [2^{(n-1)}(a_4 - a_5)(1 - 2\sin^2 \theta)]r^2 + [b_4 \cos \theta + b_5 \sin \theta]r \right\}$$

for $n = 4, 6, 8, 10, 12, 14, \dots$

$$\frac{\partial^2 \mathcal{P}}{\partial \theta^n} = \mathcal{P}_{(n\theta)} = (-1)^{\frac{n-3}{2}} \left\{ [2^n(a_4 - a_5)\sin \theta \cos \theta]r^2 + [b_4 \sin \theta - b_5 \cos \theta]r \right\}$$

for $n = 3, 5, 7, 9, 11, 13, \dots$

Now,

$$\mathcal{P}(0,0) = 0$$

$$\mathcal{P}_{(r)}(0,0) = \mathcal{B}(0) = b_3 + b_4$$

$$\mathcal{P}_{(2r)}(0,0) = 2\alpha(0) = 2(a_3 + a_4)$$

$$\mathcal{P}_{(3r)}(0,0) = \mathcal{P}_{(4r)}(0) = \dots = \mathcal{P}_{(nr)}(0) = 0$$

$$\mathcal{P}_{(\theta)}(0,0) = \mathcal{P}_{(2\theta)}(0,0) = \dots = \mathcal{P}_{(n\theta)}(0,0) = 0$$

Computing Mixed Partial Derivatives of $\mathcal{P}(r, \theta)$

$$\frac{\partial^2 \mathcal{P}}{\partial r \partial \theta} = \mathcal{P}_{(r,\theta)} = 2[2a_1\theta + a_2 - (a_4 - a_5)2\cos \theta \sin \theta]r + [2b_1\theta + b_2 - b_4 \sin \theta + b_5 \cos \theta]$$

$$\frac{\partial^3 \mathcal{P}}{\partial r \partial \theta^2} = \mathcal{P}_{(r,2\theta)} = 2[2a_1 - 2(a_4 - a_5)(1 - 2\sin^2 \theta)]r + [2b_1 - b_4 \cos \theta - b_5 \sin \theta]$$

$$\frac{\partial^{n+1} \rho}{\partial r \partial \theta^n} = \rho_{(r,n\theta)} = (-1)^{\frac{n-3}{2}} \left\{ 2[2^n(a_4 - a_5) \sin \theta \cos \theta]r + [b_4 \sin \theta - b_5 \cos \theta] \right\}$$

for $n = 3, 5, 7, 9, 11, \dots$

$$\frac{\partial^{n+1} \rho}{\partial r \partial \theta^n} = \rho_{(r,n\theta)} = (-1)^{\frac{n-4}{2}} \left\{ 2[2^{(n-1)}(a_4 - a_5)(1 - 2 \sin^2 \theta)]r + [b_4 \cos \theta + b_5 \sin \theta] \right\}$$

for $n = 4, 6, 8, 10, 12, \dots$

$$\frac{\partial^3 \rho}{\partial r^2 \partial \theta} = \rho_{(2r,\theta)} = 2[2a_1\theta + a_2 - (a_4 - a_5)2 \cos \theta \sin \theta]$$

$$\frac{\partial^4 \rho}{\partial r^2 \partial \theta^2} = \rho_{(2r,2\theta)} = 2[2a_1 - 2(a_4 - a_5)(1 - 2 \sin^2 \theta)]$$

$$\frac{\partial^{n+2} \rho}{\partial r^2 \partial \theta^n} = \rho_{(2r,n\theta)} = (-1)^{\frac{n-3}{2}} \left\{ 2[2^n(a_4 - a_5) \sin \theta \cos \theta] \right\}$$

for $n = 3, 5, 7, 9, 11, \dots$

$$\frac{\partial^{n+2} \rho}{\partial r^2 \partial \theta^n} = \rho_{(2r,n\theta)} = (-1)^{\frac{n-4}{2}} \left\{ 2[2^{(n-1)}(a_4 - a_5)(1 - 2 \sin^2 \theta)] \right\}$$

for $n = 4, 6, 8, 10, 12, \dots$

$$\frac{\partial^{n+m} \rho}{\partial r^m \partial \theta^n} = \rho_{(mr,n\theta)} = 0 \quad \text{for } m \geq 3, \quad n.$$

Finally:

$$\rho_{(r,\theta)}(0,0) = b_2 + b_5$$

$$\rho_{(r,2\theta)}(0,0) = 2b_1 - b_4$$

$$P_{(r,n\theta)}(0,0) = (-1)^{\frac{n-3}{2}} (-b_5) \quad \text{for } n = 3, 5, 7, 9, 11, \dots$$

$$P_{(r,n\theta)}(0,0) = (-1)^{\frac{n-4}{2}} (b_4) \quad \text{for } n = 4, 6, 8, 10, 12, \dots$$

$$P_{(2r,\theta)}(0,0) = 2a_2$$

$$P_{(2r,2\theta)}(0,0) = 2[2a_1 - 2(a_4 - a_5)] = 4(a_1 - a_4 + a_5)$$

$$P_{(2r,n\theta)}(0,0) = 0 \quad \text{for } n = 3, 5, 7, 9, 11, \dots$$

$$P_{(2r,n\theta)}(0,0) = (-1)^{\frac{n-4}{2}} \{2[2^{(n-1)}(a_4 - a_5)]\} = (-1)^{\frac{n-4}{2}} 2^n (a_4 - a_5)$$

for $n = 4, 6, 8, 10, 12, \dots$

$$P_{(mr,n\theta)} = 0 \quad \text{for } m \geq 3, \text{ all } n.$$

Summary

$$\rightarrow P(0,0) = 0$$

$$\rightarrow P_{(r)}(0,0) = b_3 + b_4$$

$$\rightarrow P_{(2r)}(0,0) = 2(a_3 + a_4)$$

$$\rightarrow P_{(mr)}(0,0) = 0 \quad \text{for } m \geq 3$$

$$\rightarrow P_{(n\theta)}(0,0) = 0 \quad \text{for } n \geq 1$$

$$\begin{aligned}
&\rightarrow P_{(r,\theta)}(0,0) = b_2 + b_5 \\
&\rightarrow P_{(r,2\theta)}(0,0) = 2b_1 - b_4 \\
&\rightarrow P_{(r,n\theta)}(0,0) = (-1)^{\frac{n-1}{2}} (b_5) \text{ for } n=3,5,7,9,11,\dots \\
&\rightarrow P_{(r,n\theta)}(0,0) = (-1)^{\frac{n}{2}} (b_4) \text{ for } n=4,6,8,10,12,\dots \\
&\rightarrow P_{(2r,\theta)}(0,0) = 2a_2 \\
&\rightarrow P_{(2r,2\theta)}(0,0) = 4(a_1 - a_4 + a_5) = 4a_1 + (-1)^{\frac{n}{2}} 2^2(a_4 - a_5) \\
&\rightarrow P_{(2r,n\theta)}(0,0) = 0 \text{ for } n=3,5,7,9,11,\dots \\
&\rightarrow P_{(2r,n\theta)}(0,0) = (-1)^{\frac{n}{2}} 2^n (a_4 - a_5) \text{ for } n=4,6,8,10,12,\dots \\
&\rightarrow P_{(mr,n\theta)}(0,0) = 0 \text{ for } m \geq 3, n \geq 1
\end{aligned}$$

Let $P_{(mr,n\theta)}(0,0) = P_{mr,n\theta}^0$

The formulas for $P_{mr,n\theta}^0$ can all be condensed into the following expression:

$$\begin{aligned}
P_{mr,n\theta}^0 = & \left\{ [2b_1 \rho_n(2) + b_2 \rho_n(1) + b_3 \rho_n(0)] \rho_m(1) \right. \\
& + [4a_1 \rho_n(2) + 2a_2 \rho_n(1) + (2a_3 + a_4 + a_5) \rho_n(0)] \rho_m(2) \\
& + \{b_4 \cos(\frac{n}{2}\pi) + b_5 \cos[(\frac{n-1}{2})\pi]\} \rho_m(1) \\
& \left. + \{2^n (a_4 - a_5) \cos(\frac{n}{2}\pi)\} \rho_m(2) \right] \prod_{j=3}^{\infty} [1 - \rho_m(j)] \} [1 - \rho_m(0)]
\end{aligned}$$

$$\text{Where } \left. \begin{aligned} \rho_k(j) &= 1 & k=j \\ \rho_k(j) &= 0 & k \neq j \end{aligned} \right\} \begin{array}{l} \text{DISCRETE UNIT} \\ \text{STEP FUNCTION} \end{array}$$



Now consider the partial derivatives of

$$g(r, \theta) = e^{P(r, \theta)}$$

with respect to r and θ .

Define
$$\frac{\partial^{m+n} g}{\partial r^m \partial \theta^n} = g_{mr, n\theta}$$

$$g = e^P$$

$$g_r = e^P P_r = g P_r$$

$$g_{2r} = g_r P_r + g P_{2r}$$

$$\begin{aligned} g_{3r} &= g_{2r} P_r + g_r P_{2r} + g_r P_{2r} + g P_{3r} \\ &= g_{2r} P_r + 2 g_r P_{2r} + g P_{3r} \end{aligned}$$

$$\begin{aligned} g_{4r} &= g_{3r} P_r + g_{2r} P_{2r} + 2 g_{2r} P_{2r} + 2 g_r P_{3r} + g_r P_{3r} + g P_{4r} \\ &= g_{3r} P_r + 3 g_{2r} P_{2r} + 3 g_r P_{3r} + g P_{4r} \end{aligned}$$

$$\begin{aligned}
g_{5r} &= g_{4r} p_r + g_{3r} p_{2r} + 3g_{3r} p_{2r} + 3g_{2r} p_{3r} \\
&\quad + 3g_{2r} p_{3r} + 3g_r p_{4r} + g_r p_{4r} + g p_{5r} \\
&= g_{4r} p_r + 4g_{3r} p_{2r} + 6g_{2r} p_{3r} + 4g_r p_{4r} + g p_{5r} \\
&\vdots \\
g_{mr} &= g_{(m-1)r} p_r + \binom{m-1}{1} g_{(m-2)r} p_{2r} + \binom{m-1}{2} g_{(m-3)r} p_{3r} \\
&\quad + \dots + \binom{m-1}{m-2} g_r p_{(m-1)r} + \binom{m-1}{m-1} g p_{mr}
\end{aligned}$$

for $m \geq 1$

But $p_{mr}^\circ = 0$ for $m \geq 3$. Therefore,

$$g_{mr}^\circ = g_{(m-1)r}^\circ p_r^\circ + \binom{m-1}{1} g_{(m-2)r}^\circ p_{2r}^\circ \quad \text{for } m \geq 1$$

Similarly,

$$\begin{aligned}
g_{n\theta} &= g_{(n-1)\theta} p_\theta + \binom{n-1}{1} g_{(n-2)\theta} p_{2\theta} + \binom{n-1}{2} g_{(n-3)\theta} p_{3\theta} \\
&\quad + \dots + \binom{n-1}{n-2} g_\theta p_{(n-1)\theta} + \binom{n-1}{n-1} g p_{n\theta}
\end{aligned}$$

But $p_{n\theta}^\circ = 0$ for $n \geq 1$

Therefore, $g_{n\theta}^\circ = 0$ for $n \geq 1$

To obtain mixed partial derivatives, consider,

$$g_r = g p_r$$

$$g_{r,\theta} = g_\theta p_r + g p_{r,\theta}$$

$$\begin{aligned} g_{r,2\theta} &= g_{2\theta} p_r + g_\theta p_{r,\theta} + g_\theta p_{r,\theta} + g p_{r,2\theta} \\ &= g_{2\theta} p_r + 2 g_\theta p_{r,\theta} + g p_{r,2\theta} \end{aligned}$$

$$\begin{aligned} g_{r,3\theta} &= g_{3\theta} p_r + g_{2\theta} p_{r,\theta} + 2 g_{2\theta} p_{r,\theta} + 2 g_\theta p_{r,2\theta} \\ &\quad + g_\theta p_{r,2\theta} + g p_{r,3\theta} \\ &= g_{3\theta} p_r + 3 g_{2\theta} p_{r,\theta} + 3 g_\theta p_{r,2\theta} + g p_{r,3\theta} \end{aligned}$$

$$\begin{aligned} g_{r,n\theta} &= g_{n\theta} p_r + \binom{n}{1} g_{(n-1)\theta} p_{r,\theta} + \binom{n}{2} g_{(n-2)\theta} p_{r,2\theta} \\ &\quad + \dots + \binom{n}{n-1} g_\theta p_{r,(n-1)\theta} + \binom{n}{n} g p_{r,n\theta} \end{aligned}$$

for $n \geq 1$

Therefore,

$$g_{r,n\theta}^\circ = \sum_{k=0}^n \binom{n}{k} g_{(n-k)\theta}^\circ p_{r,k\theta}^\circ \quad \text{for } n \geq 1$$

$$\text{also, } g_{2r} = [g_r p_r] + [g p_{2r}]$$

$$g_{2r,\theta} = [g_{r,\theta} p_r + g_r p_{r,\theta}] + [g_\theta p_{2r} + g p_{2r,\theta}]$$

$$\begin{aligned}
g_{2r,2\theta} &= [g_{r,2\theta} p_r + g_{r,\theta} p_{r,\theta} + g_{r,\theta} p_{r,\theta} + g_r p_{r,2\theta}] \\
&\quad + [g_{2\theta} p_{2r} + g_{\theta} p_{2r,\theta} + g_{\theta} p_{2r,\theta} + g p_{2r,2\theta}] \\
&= [g_{r,2\theta} p_r + 2g_{r,\theta} p_{r,\theta} + g_r p_{r,2\theta}] \\
&\quad + [g_{2\theta} p_{2r} + 2g_{\theta} p_{2r,\theta} + g p_{2r,2\theta}]
\end{aligned}$$

$$\begin{aligned}
g_{2r,n\theta} &= [g_{r,n\theta} p_r + \binom{n}{1} g_{r,(n-1)\theta} p_{r,\theta} + \binom{n}{2} g_{r,(n-2)\theta} p_{r,2\theta} + \dots \\
&\quad + \binom{n}{n-1} g_{r,\theta} p_{r,(n-1)\theta} + \binom{n}{n} g_r p_{r,n\theta}] \\
&\quad + [g_{n\theta} p_{2r} + \binom{n}{1} g_{(n-1)\theta} p_{2r,\theta} + \binom{n}{2} g_{(n-2)\theta} p_{2r,2\theta} + \dots \\
&\quad + \binom{n}{n-1} g_{\theta} p_{2r,(n-1)\theta} + \binom{n}{n} g p_{2r,n\theta}] \quad \text{for } n \geq 1
\end{aligned}$$

Therefore,

$$g_{2r,n\theta} = \sum_{k=0}^n \binom{n}{k} g_{r,(n-k)\theta} p_{r,k\theta} + \sum_{k=0}^n \binom{n}{k} g_{(n-k)\theta} p_{2r,k\theta}$$

for $n \geq 1$

Finally,

$$g_{2r,n\theta}^{\circ} = \sum_{k=0}^n \binom{n}{k} [g_{r,(n-k)\theta}^{\circ} p_{r,k\theta}^{\circ} + g_{(n-k)\theta}^{\circ} p_{2r,k\theta}^{\circ}]$$

for $n \geq 1$

In general,

$$g_{mr}^{\circ} = g_{(m-1)r}^{\circ} p_r^{\circ} + \binom{m-1}{1} g_{(m-2)r}^{\circ} p_{2r}^{\circ}$$

for $n \geq 1$

From which

$$g_{mr, n\theta}^{\circ} = \sum_{k=0}^n \binom{n}{k} [g_{(m-1)r, (n-k)\theta}^{\circ} p_{r, k\theta}^{\circ} + \binom{m-1}{1} g_{(m-2)r, (n-k)\theta}^{\circ} p_{2r, k\theta}^{\circ}]$$

for $m \geq 1, n \geq 1$

Summary

(1) $g^{\circ} = 1$ ($m = n = 0$)

(2) $g_{mr}^{\circ} = g_{(m-1)r}^{\circ} p_r^{\circ} + \binom{m-1}{1} g_{(m-2)r}^{\circ} p_{2r}^{\circ}$ for $m \geq 1$

(3) $g_{n\theta}^{\circ} = 0$ for $n \geq 1$

(4) $g_{(mr, n\theta)}^{\circ} = \sum_{k=0}^n \binom{n}{k} \{ g_{(m-1)r, (n-k)\theta}^{\circ} p_{r, k\theta}^{\circ} + \binom{m-1}{1} g_{(m-2)r, (n-k)\theta}^{\circ} p_{2r, k\theta}^{\circ} \}$
for $m \geq 1, n \geq 1$

Where

$$p_{mr, n\theta}^{\circ} = \left\{ [2b_1 a_n(2) + b_2 a_n(1) + b_3 a_n(0)] a_m(1) + [4a_1 a_n(2) + 2a_2 a_n(1) + (2a_3 - \frac{1}{2\sigma_x^2} - \frac{1}{2\sigma_y^2}) a_n(0)] a_m(2) + [H_x \cos(\frac{\pi}{2}\pi) + H_y \cos[(\frac{\pi-1}{2})\pi]] a_m(1) + [2^n (\frac{1}{2\sigma_y^2} - \frac{1}{2\sigma_x^2}) \cos(\frac{\pi}{2}\pi)] a_m(2) \right. \\ \left. \prod_{j=3}^{\infty} [1 - a_m(j)] \right\} [1 - a_m(0)]$$

and

$$\Delta_k(j) = 1 \quad \text{for } k=j$$

$$\Delta_k(j) = 0 \quad \text{for } k \neq j$$

The expressions for $g_{m,r,n\theta}^\circ$ can be condensed as follows:

I. Notice (4) reproduces (2) for $m=0$

$$g_{m,r,n\theta}^\circ = \Delta_m(0) \Delta_n(0) + \left[\sum_{k=0}^n \binom{n}{k} \{ g_{(m-1)r,(n-k)\theta}^\circ P_{r,k\theta}^\circ + \binom{m-1}{1} g_{(m-2)r,(n-k)\theta}^\circ P_{2r,k\theta}^\circ \} \right] [1 - \Delta_m(0)]$$

for $m, n \geq 0$

Test:

For $m=0, n=0$

$$g^\circ = 1 \quad \text{OK.}$$

For $m=0, n \neq 0$

$$g_{n\theta}^\circ = 0 \quad \text{OK.}$$

For $m \neq 0, n=0$

$$g_{mr}^\circ = g_{(m-1)r}^\circ P_r^\circ + \binom{m-1}{1} g_{(m-2)r}^\circ P_{2r}^\circ \quad \text{OK.}$$

For $m \neq 0, n \neq 0$

$$g_{m,r,n\theta}^\circ = \sum_{k=0}^n \binom{n}{k} \{ g_{(m-1)r,(n-k)\theta}^\circ P_{r,k\theta}^\circ + \binom{m-1}{1} g_{(m-2)r,(n-k)\theta}^\circ P_{2r,k\theta}^\circ \} \quad \text{OK.}$$

Now, the Maclaurin expansion of $g(r, \theta)$ is given by,

$$g(r, \theta) = \sum_{i=0}^{\infty} \frac{[r \frac{\partial}{\partial r} + \theta \frac{\partial}{\partial \theta}]^i g(0,0)}{i!}$$

$$\begin{aligned} \text{But } [r \frac{\partial}{\partial r} + \theta \frac{\partial}{\partial \theta}]^i g(0,0) &= g_{ir}^0 r^i + \binom{i}{1} g_{(i-1)r, \theta}^0 r^{i-1} \theta \\ &\quad + \binom{i}{2} g_{(i-2)r, 2\theta}^0 r^{i-2} \theta^2 + \dots \\ &\quad + \binom{i}{i-1} g_{r, (i-1)\theta}^0 r \theta^{i-1} + \binom{i}{i} g_{i\theta}^0 \theta^i \\ &= \sum_{j=0}^i \binom{i}{j} g_{(i-j)r, j\theta}^0 r^{(i-j)} \theta^j. \end{aligned}$$

Consequently,

$$g(r, \theta) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} g_{(i-j)r, j\theta}^0 r^{(i-j)} \theta^j \right]$$

Now, $\int_0^{2\pi} \int_0^{\infty} g(r, \theta) r dr d\theta$ can be computed.

For practical and obvious reasons, the integration along the r coordinate will be performed from 0 to r_0 , where r_0 is some limiting value radius where the regressed equation for $f_h(r, \theta)$ applies. Also, the integration on θ will be carried out from 0 to θ_0 where θ_0 is one full cycle (360° angle). Both r_0 and θ_0 must be smaller than unity, to speed up convergence. This is accomplished by proper scaling of the coordinate system.

Therefore

$$\begin{aligned} &\int_0^{\theta_0} \int_0^{r_0} g(r, \theta) r dr d\theta = \\ &= \int_0^{\theta_0} \int_0^{r_0} \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} g_{(i-j)r, j\theta}^0 r^{(i-j)} \theta^j \right] r dr d\theta \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\theta_0} \int_0^{r_0} \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} g_{(i-j)r, j\theta}^{\circ} r^{i-(j-1)} \theta^j \right] dr d\theta \\
&= \int_0^{\theta_0} \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{i=j}^{\infty} \binom{i}{j} g_{(i-j)r, j\theta}^{\circ} \frac{r_0^{i-(j-2)} \theta^{j+1}}{i-(j-2)} \right] d\theta \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} g_{(i-j)r, j\theta}^{\circ} \frac{r_0^{i-(j-2)} \theta^{j+1}}{[i-(j-2)](j+1)} \right]
\end{aligned}$$

Observe that, consistent with notation, $m = i-j$ and $n = j$.

Therefore,

$$\begin{aligned}
&\int_0^{\theta_0} \int_0^{r_0} g(r, \theta) r dr d\theta = \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} \frac{r_0^{i-(j-2)} \theta^{j+1}}{[i-(j-2)](j+1)} g_{(i-j)r, j\theta}^{\circ} \right] \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} \frac{r_0^{i-(j-2)} \theta^{j+1}}{[i-(j-2)](j+1)} g_{(i-j)r, j\theta}^{\circ} \right] \\
&+ \sum_{i=1}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} \frac{r_0^{i-(j-2)} \theta^{j+1}}{[i-(j-2)](j+1)} g_{(i-j)r, j\theta}^{\circ} \right] \\
&= \frac{1}{0!} \left[\binom{0}{0} \frac{r_0^2 \theta^1}{(2)(1)} g^{\circ} + \sum_{i=1}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^i \binom{i}{j} \frac{r_0^{i-(j-2)} \theta^{j+1}}{[i-(j-2)](j+1)} g_{(i-j)r, j\theta}^{\circ} \right] \right]
\end{aligned}$$

But $g^{\circ} = 1$ and $g_{n\theta}^{\circ} = 0 \Rightarrow$ for $n = 0 (i-j)$

Then,
$$\int_0^{\theta_0} \int_0^{r_0} g(r, \theta) r dr d\theta =$$
$$= \frac{\theta_0 r_0^2}{2} + \sum_{i=1}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^{i-1} \binom{i}{j} \frac{r_0^{i-(j-2)} \theta_0^{j+1}}{[i-(j-2)](j+1)} g_{(i-j)r, j\theta}^{\circ} \right]$$

Finally:

$$\int_0^{\theta_0} \int_0^{r_0} \rho_T(r, \theta) f(r, \theta) r dr d\theta = \frac{e^{\{C - \frac{1}{2} H x^2 - \frac{1}{2} H y^2\}}}{2\pi\sigma_x\sigma_y} \left\{ \frac{\theta_0 r_0^2}{2} + \sum_{i=1}^{\infty} \frac{1}{i!} \left[\sum_{j=0}^{i-1} \binom{i}{j} \frac{r_0^{i-(j-2)} \theta_0^{j+1}}{[i-(j-2)](j+1)} g_{(i-j)r, j\theta}^{\circ} \right] \right\}$$

Where

$$g_{mr, n\theta}^{\circ} = \left[\sum_{k=0}^n \binom{n}{k} \left\{ g_{(m-1)r, (n-k)\theta}^{\circ} \rho_{r, k\theta}^{\circ} + \right. \right. \\ \left. \left. + \binom{m-1}{1} g_{(m-2)r, (n-k)\theta}^{\circ} \rho_{2r, k\theta}^{\circ} \right\} \right] [1 - \delta_m(0)] + \delta_m(0) \delta_n(0)$$

for $m \geq 0, n \geq 0$

$$\rho_{mr,n0}^0 = [2b_1 \Delta_n(2) + b_2 \Delta_n(1) + b_3 \Delta_n(0)] \Delta_m(1)$$

$$+ [4a_1 \Delta_n(2) + 2a_2 \Delta_n(1) + (2a_3 - \frac{1}{2\sigma_x^2} - \frac{1}{2\sigma_y^2}) \Delta_n(0)] \Delta_m(2)$$

$$+ \{H_x \cos(\frac{\pi}{2}\pi) + H_y \cos[(\frac{\pi-1}{2})\pi]\} \Delta_m(1)$$

$$+ \{2^{\pi}(\frac{1}{2\sigma_y^2} - \frac{1}{2\sigma_x^2}) \cos(\frac{\pi}{2}\pi)\} \Delta_m(2)$$

$$\text{for } m = 1, 2; n \geq 0$$

$$\text{and } \Delta_k(j) = 1 \text{ for } k = j$$

$$\Delta_k(j) = 0 \text{ for } k \neq j$$

APPENDIX II

RESULTS OF REGRESSION ANALYSIS

Curves of the form given in equations 21 and 22 were fit to numerically defined damage functions for weapon-target case 152.

The damage functions used are listed in Table II-I along with the corresponding goodness-of-fit ratios (GOF) and sums of the squares of the deviations of the curves from the observed data ($\Sigma(y-y)^2$). The goodness-of-fit ratio is defined as the ratio of the variance explained by the explicit relationship to the variance of the numerically defined function (the number one represents perfect correlation).

For high height of burst (HOB) the values from the numerical functions are very small and almost randomly scattered. Curves fit to such data will have small $\Sigma(y-y)^2$ since the deviations from the zero base are small to begin with; the same curves may have very poor goodness-of-fit since they explain very little of the variance present in the data. It is possible that there is no relationship between such data points (complete randomness); small values of $\Sigma(y-y)^2$ are still possible.

The ninety-degree elevation angle of attack (ANG) cases can be fit with symmetrical surfaces. The goodness-of-fit ratios for these cases are larger than those for lower elevation angles of attack.

This regression analysis was based on 3060 data points. The sums of the squares of the deviations ($\Sigma(y-y)^2$) in Table II-1 are over that number of points.

TABLE II-I. SUMMARY OF RESULTS OF REGRESSION ANALYSIS
(Weapon-Target Case 152)

Terminal Conditions			Regression Curves	
HOB	ANG	VEL	GOF	$\Sigma(y-y)^2$
1.40	45.00	700.00	0.933629	0.39671E 02
1.40	45.00	1000.00	0.874630	0.75235E 02
1.40	45.00	1300.00	0.955912	0.26579E 02
1.90	70.00	700.00	0.958644	0.24414E 02
1.90	70.00	1000.00	0.962145	0.22360E 02
1.90	70.00	1300.00	0.960273	0.23510E 02
2.10	90.00	700.00	0.997513	0.15603E 01
2.10	90.00	1000.00	0.995336	0.26659E 01
2.10	90.00	1300.00	0.994256	0.31494E 01
20.00	45.00	700.00	0.795667	0.13674E 02
20.00	45.00	1000.00	0.786459	0.16873E 02
20.00	45.00	1300.00	0.777634	0.20675E 02
20.00	70.00	700.00	0.767283	0.15696E 02
20.00	70.00	1000.00	0.821736	0.16674E 02
20.00	70.00	1300.00	0.826441	0.21226E 02
20.00	90.00	700.00	0.807875	0.29011E 02
20.00	90.00	1000.00	0.837392	0.27062E 02
20.00	90.00	1300.00	0.892173	0.23002E 02
40.00	45.00	700.00	0.663889	0.14383E 01
40.00	45.00	1000.00	0.653946	0.18524E 01
40.00	45.00	1300.00	0.650797	0.23304E 01
40.00	70.00	700.00	0.618090	0.17297E 01
40.00	70.00	1000.00	0.669085	0.21498E 01
40.00	70.00	1300.00	0.709352	0.26624E 01
40.00	90.00	700.00	0.972755	0.20575E 01
40.00	90.00	1000.00	0.967214	0.28898E 01
40.00	90.00	1300.00	0.961912	0.38102E 01
80.00	45.00	700.00	0.571082	0.75003E 01
80.00	45.00	1000.00	0.539971	0.10604E 00
80.00	45.00	1300.00	0.272525	0.21975E 00
80.00	70.00	700.00	0.513242	0.10193E 00

TABLE II-I (CONCLUDED)

HOB	ANG	VEL	GOF	$\Sigma(y-y)^2$
80.00	70.00	1000.00	0.544083	0.13480E 00
80.00	70.00	1300.00	0.604326	0.17633E 00
80.00	90.00	700.00	0.945677	0.10548E 01
80.00	90.00	1000.00	0.950072	0.12186E 01
80.00	90.00	1300.00	0.956756	0.13884E 01
120.00	45.00	700.00	0.514104	0.11593E 01
120.00	45.00	1000.00	0.435045	0.18436E 01
120.00	45.00	1300.00	0.429930	0.23478E 01
120.00	70.00	700.00	0.521283	0.14438E 01
120.00	70.00	700.00	0.521283	0.14438E 01
120.00	70.00	1000.00	0.529390	0.18597E 01
120.00	70.00	1300.00	0.549753	0.24629E 01
120.00	90.00	700.00	0.924258	0.56311E 00
120.00	90.00	1000.00	0.928711	0.68378E 00
120.00	90.00	1300.00	0.932487	0.81306E 00

These results and the actual regression coefficients are available from the researchers in punch card coded form.

APPENDIX III

ACCELERATION ALGORITHM TO MINIMIZE A CONVEX OBJECTIVE FUNCTION SUBJECT TO LINEAR CONSTRAINTS

ABSTRACT

An acceleration algorithm to minimize a convex objective function (one that is never underestimated by a linear interpolation between two points) subject to linear constraints is presented in detail and an example problem is given. It reduces the number of changes in the state set and the number of iterations required by the general differential algorithm for linear constraints in converging to the minimum. It can also be used to maximize a concave objective function (one whose negative is convex) subject to linear constraints.

INTRODUCTION

An objective function $y(\bar{x})$ is convex if it is never underestimated by a linear interpolation between two points, \bar{x}_1 and \bar{x}_2 . That is, for every α satisfying

$$0 < \alpha < 1 \quad (1)$$

it is true that

$$y(\alpha \bar{x}_1 + [1-\alpha] \bar{x}_2) \leq \alpha y(\bar{x}_1) + [1-\alpha] y(\bar{x}_2). \quad (2)$$

The adjective "strictly" is added if the function is never exactly equal to the linear interpolation between the two points, i.e., if the \leq sign is replaced by the $<$ sign. A concave function is one whose negative is convex (Zukhovitskiy and Avdeyeva, Reference 2).

If a non-linear function is to be minimized subject to linear constraints, i.e., a problem of the form:

$$\text{Min. } y(\bar{x}), \text{ where } \bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad (3)$$

subject to,

$$x_n \geq 0 \text{ for } n = 1, \dots, N$$

and

$$\sum_{n=1}^N a_{kn} x_n \geq b_k; \quad k = 1, \dots, K, \quad (4)$$

The following differential algorithm, given by Wilde and Beightler can be used (Reference 1, pp. 65-66):

1. Let v_i be the most negative decision derivative and v_h the most positive decision derivative for which the corresponding decision variable d_h is positive.
2. If there are no negative v_m , set v_i to zero. If all positive v_n have corresponding d_n equal to zero, set v_h to zero.
3. If both $v_i = 0$ and $v_h = 0$, a stationary point has been found. Notice that when both v_i and v_h are zero, the Kuhn-Tucker necessary conditions, i.e., non-negativity and complementary slackness, are satisfied.

4. If either v_i or v_h is not zero, compute

$$V = v_i + v_h \quad (5)$$

5. If $V \leq 0$, increase d_i . If $V > 0$, decrease d_h , holding all other decision variables constant, but permitting the state variables to readjust, until
6. I. Some state variable, say s_p becomes zero, or
 II. v_r becomes zero; where $r = i$ if $V \leq 0$, and $r = h$ if $V > 0$, or
 III. d_h becomes zero.
7. In case (I), d_r replaces s_p in the state set; in cases (II) and (III) no change in the set of states occurs.
8. Clearly, for any change, δd_r , the value of the objective function y always decreases since

$$\Delta y = \int v_r \delta d_r < 0. \quad (6)$$

The greatest difficulty in the application of the preceding algorithm occurs in step 7, case (I), when the variables d_r and s_p are interchanged, i.e., when a change in the state set becomes necessary, because a new set of constrained derivatives must be computed as follows:

$$v_r = \frac{\delta y}{\delta d_r} = \frac{\partial y}{\partial d_r} - \sum_{k=1}^K \alpha_{kr} \frac{\partial y}{\partial s_k}; \quad r = 1, \dots, N \quad (7)$$

where, $v_r = \frac{\delta y}{\delta d_r}$ is the constrained partial derivative of the objective function y with respect to the decision variable d_r ;

$\frac{\partial y}{\partial d_r}$ is the unconstrained partial derivative of the objective function y with respect to the decision variable d_r ;

$\frac{\partial y}{\partial s_k}$ is the unconstrained partial derivative of the objective function y with respect to the state variable s_k ; and

α_{kr} is a constant coefficient obtained from the following linear equation relating the state variable s_k to the decision variables d_n , $n=1, \dots, N$,

$$s_k = \theta_k - \sum_{n=1}^N \alpha_{kn} d_n, \quad k = 1, \dots, K; \quad (8)$$

θ_k is a constant term in the expression for s_k .

Acceleration Algorithm

When the objective function is convex, the following minimization algorithm can be used to reduce the number of changes in the state set and to decrease the number of iterations required for convergence. The same algorithm can also be used to maximize concave objectives subject to linear constraints.

- I. Compute minimum point \bar{x}^0 for unconstrained objective function.
- II. Check the constraint set to ascertain:
 1. If all the constraints are loose at \bar{x}^0 , the minimizing policy is $x^* = \bar{x}^0$. STOP.
 2. If one or more constraints are tight or violated at \bar{x}^0 , CONTINUE.
- III.
 1. Let the slack variables of the constraints that are tight and/or violated at \bar{x}^0 be in the decision set d .
 2. Let the slack variables for the constraints that are loose at \bar{x}^0 be in the state set s .
 3. Complete the state and decision sets with the structural variables of the problem.
- IV. Set the decision variables to zero,
$$\bar{d} = \bar{0}. \quad (9)$$
- V. Use the differential algorithm for a non-linear objective function subject to linear constraints to find the minimizing policy and verify that the point obtained satisfies the sufficiency conditions for a minimum.

The algorithm guarantees that no changes in the state set will occur provided that all the loose constraints at \bar{x}^0 remain loose at x^* . Since the decision variables are set to zero initially, they can only increase in value to satisfy non-negativity. That means that all the slack variables for the tight and violated constraints at \bar{x}^0 will be zero at the beginning of the algorithm, thus making these constraints tight initially. Since the decisions are manipulated at will and cannot go negative, the initially tight constraints can either remain tight, if the corresponding slacks remain at zero, or go loose, if the corresponding slacks increase, and unless one or more of the loose constraints at \bar{x}^0 is tight at x^* , no changes in the state set will be required and the number of iterations for convergence to the minimum is considerably reduced.

Illustrative Problem and Conclusions

Application of the acceleration algorithm will be illustrated with the following problem involving a quadratic (convex) objective function:

$$\text{Min. } y = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 \quad (10)$$

subject to

$$x_1, x_2 \geq 0; \quad (11)$$

$$3x_1 + 4x_2 \leq 6; \quad (12)$$

and

$$-x_1 + 4x_2 \leq 2. \quad (13)$$

Step I. To compute the minimum point for the unconstrained objective function set,

$$\frac{\partial y}{\partial x_1} = 4x_1 - 2x_2 - 6 = 0, \quad (14)$$

$$\frac{\partial y}{\partial x_2} = 4x_2 - 2x_1 = 0,$$

from which

$$\bar{x}^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \quad (15)$$

The differential quadratic form is positive definite, that is,

$$\begin{aligned} \partial \bar{x}^T \bar{H} \partial \bar{x} &= (\partial x_1, \partial x_2) \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \partial x_1 \\ \partial x_2 \end{pmatrix} \\ &= 4 \left(\partial x_1 - \frac{1}{2} \partial x_2 \right)^2 + 3 \partial x_2^2 > 0, \end{aligned} \quad (16)$$

for $\partial x_1, \partial x_2 \neq 0$.

This guarantees that \bar{x}^0 is in fact a global minimum.

Step II. A check of the constraint set reveals that constraint (12) is violated and constraint (13) is tight at \bar{x}^0 .

Step III. Therefore, after the introduction of slack variables, the problem becomes:

$$\text{Min. } y = 2s_1^2 - 2s_1s_2 + 2s_2^2 - 6s_1 \quad (17)$$

$$\text{subject to } s_1, s_2, d_1, d_2 \geq 0;$$

$$3s_1 + 4s_2 + d_1 = 6; \quad (18)$$

$$\text{and } -s_1 + 4s_2 + d_2 = 2,$$

$$\text{with } d_1 = x_3; \quad d_2 = x_4; \quad s_1 = x_1; \quad s_2 = x_2. \quad (19)$$

Solving for s_1 and s_2 in terms of d_1 and d_2 obtain

$$s_1 = 1 - \left(\frac{1}{4} d_1 - \frac{1}{4} d_2 \right); \quad (20)$$

$$s_2 = \frac{3}{4} - \left(\frac{1}{16} d_1 + \frac{3}{16} d_2 \right). \quad (21)$$

Because of the form of equation (8),

$$\left. \begin{aligned} \beta_1 &= 1; \quad \alpha_{11} = \frac{1}{4}; \quad \alpha_{12} = -\frac{1}{4}; \\ \beta_2 &= \frac{3}{4}; \quad \alpha_{21} = \frac{1}{16}; \quad \alpha_{22} = \frac{3}{16}. \end{aligned} \right\} \quad (22)$$

Therefore, from equation (7),

$$\begin{aligned} v_1 &= \frac{\delta y}{\delta d_1} = \frac{\partial y}{\partial d_1} - \left(\alpha_{11} \frac{\partial y}{\partial s_1} + \alpha_{21} \frac{\partial y}{\partial s_2} \right) \\ &= \frac{1}{64} (52 + 13 d_1 - 17 d_2); \end{aligned} \quad (23)$$

$$\begin{aligned} \text{and } v_2 &= \frac{\delta y}{\delta d_2} = \frac{\partial y}{\partial d_2} - \left(\alpha_{12} \frac{\partial y}{\partial s_1} + \alpha_{22} \frac{\partial y}{\partial s_2} \right) \\ &= \frac{1}{64} (-68 - 17 d_1 + 37 d_2). \end{aligned} \quad (24)$$

Step IV. Set $d_1 = d_2 = 0$.

Step V. (1-1) Having $d_1 = d_2 = 0$, obtain,

from equation (20), $s_1 = 1$;

from equation (21), $s_2 = \frac{3}{4}$;

from equation (23), $v_1 = \frac{13}{16}$;

from equation (24), $v_2 = -\frac{17}{64}$.

(25)

(2-1) The differential algorithm requires,

$$v_i = v_2 = -\frac{17}{64}; v_h = 0. \quad (26)$$

(3-1) Both v_i and v_h are not zero, therefore the minimum has not yet been found.

(4-1) Since $v_i = v_2 = -\frac{17}{64}$, compute

$$V = v_i + v_h = -\frac{17}{64}. \quad (27)$$

(5-1) Because $V < 0$, increase $d_1 = d_2$.

(6-1) The increase in d_2 is constrained by:

Case (I)

$$s_k + \Delta s_{k2} \geq 0 \text{ for } k = 1, \dots, K, \quad (28)$$

but

$$\Delta s_{k2} = \left(\frac{\partial s_k}{\partial d_2} \right) \Delta d_2 = -\alpha_{k2} \Delta d_2. \quad (29)$$

Therefore,

$$\Delta d_2 = \min_{\alpha_{k2} > 0} \left\{ \frac{s_k}{\alpha_{k2}} \right\} = \left(\frac{\frac{3}{4}}{\frac{3}{16}} \right) = 4. \quad (30)$$

Case (II)

$$v_2 + \frac{\partial v_2}{\partial d_2} \Delta d_2 = 0. \quad (31)$$

Therefore,

$$\Delta d_2 = \left(\frac{\frac{17}{16}}{\frac{37}{64}} \right) = \frac{68}{37}. \quad (32)$$

Case (III) Since d_2 is initially set at zero and is to be increased, this case does not apply.


From the results above, $\Delta d_2 = \frac{58}{37} < 4$. (33)

(7-1) No state change was needed, and the new values of the decision and state variables, and of the constrained derivatives are:


$$\left. \begin{aligned} d_1 &= 0 \quad \text{remains constant;} \\ d_2 &= 0 + \frac{68}{37} = \frac{68}{37}; \\ s_1 &= 1 - \left(-\frac{4}{16}\right) \frac{68}{37} = \frac{54}{37}; \\ s_2 &= \frac{3}{4} - \left(\frac{3}{16}\right) \frac{68}{37} = \frac{15}{37}; \\ v_1 &= \frac{13}{16} + \left(-\frac{17}{64}\right) \frac{68}{37} = \frac{12}{37}; \\ v_2 &= -\frac{17}{16} + \left(\frac{37}{64}\right) \frac{68}{37} = 0. \end{aligned} \right\} \quad (34)$$

The value of y at this point is $y = -\frac{198}{37}$. (35)

The results can be displayed in a tableau with the following format:

d_1	d_2	
α_{11}	α_{12}	s_1
α_{21}	α_{22}	s_2
v_1	v_2	y

Therefore,

0	$\frac{68}{37}$	
$\frac{4}{16}$	$-\frac{4}{16}$	$\frac{54}{37}$
$\frac{1}{16}$	$\frac{3}{16}$	$\frac{15}{37}$
$\frac{12}{37}$	0	$-\frac{198}{37}$

Now the differential algorithm is applied all over again.

(1-2) The new values of decision and state variables as well as of the decision derivatives are given in the tableau.

(2-2) $v_i = 0$ and $v_h = 0$ (36)

(3-2) Since both $v_i = 0$ and $v_h = 0$, the minimum has been found at

$$\bar{x}^* = \begin{matrix} x_1^* \\ x_2^* \end{matrix} = \begin{matrix} s_1^* \\ s_2^* \end{matrix} = \begin{matrix} \frac{54}{37} \\ \frac{15}{37} \end{matrix} \quad (37)$$

$$\text{and } y^*(\bar{x}^*) = \frac{198}{37}. \quad (38)$$

The problem converged to the solution in one iteration and no changes in the state set were necessary.

The same problem was solved by Wilde and Beightler (Reference 1, pp. 76-78) using the differential algorithm alone, without acceleration. Three iterations involving two changes in the state set were required to converge to the minimum.

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1. Wilde, J. D. and Beightler, C. S., Foundations of Optimization, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1967.
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<p>The work conducted at Louisiana State University on a class of problems arising in the evaluation of weapons and in the analysis of the best methods for their use is reported. The destructiveness of a specified weapon against a specified target is measured by the ratio of the number of targets destroyed to the number of weapons used.</p> <p>This report describes the methods developed for the analytical evaluation of average kill probabilities for single weapons and proposes the optimization of pattern parameters for multi-weapon attacks on discretely defined target complexes for future investigation.</p> <p>An acceleration algorithm for minimizing a convex objective function subject to linear constrained is also presented in detail.</p>		

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